Exact Resistance of a Modified Truncated Cone Resistor for Introductory Physics

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Abstract

Motivated by another paper on the topic [1], a modified truncated conical resistor problem for introductory physics is proposed to overcome the shortcomings and deficiencies in the calculation of the resistance of the standard truncated conical resistor. The geometry of the new shape allows the Riemann sum method of introductory physics to give its exact resistance. However, the proposed resistor has additional benefits: It allows the straightforward application of at least two important mathematical techniques to electricity for instructors who want to link directly a higher-level course to introductory physics. In particular, conformal mapping can also be used to compute its resistance. Alternatively, using boundary conditions for the behavior of the current on the faces of the modified resistor, one can solve the boundary value problem for the potential and, hence, derive the resistance of the resistor this way.

1 Introduction & Motivation

A good fraction of physics textbooks used for the introductory physics sequence course include the truncated conical resistor as an end-of-chapter problem[1]. In a paper [1] written by Romano and Price, the authors have

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[1] Over time, the trends have changed slightly. To the best of our recollection, the conical resistor used to be a standard problem in most texts. These days, some texts include it, and some do not. Among the texts that we have handy, the ones that include the conical resistor are given in Ref. [4], while the ones that do not are given in Ref. [5]. Particularly,

questioned the applicability of elementary methods to the truncated cone resistor. They have explicitly discussed the hidden issues behind the anticipated method of calculation. Of course, their concern should not be unique to the truncated cone resistor. The introductory physics textbooks are full of problems whose expected solutions, when dissected carefully, are plagued by serious, inherent deficiencies and subtleties. If accuracy is demanded, then most objects in the introductory electricity and magnetism course should be abandoned. However, in that case, we would not have simple calculations for parallel plate capacitors, cylindrical capacitors, solenoids, toroids, and many other objects for which are used as tools to teach students the fundamental ideas. Hence, avoiding all of the problems which have conceptual deficiencies at a deeper level is not a reasonable or even practical position for the introductory course. Instead, behind any calculation in introductory physics, the prevailing attitude should be: *Assume all necessary conditions that ensure the applicability of the simple method as a good approximation*, then perform the calculation. To avoid repetition during the introductory course, the first part of the previous statement is typically omitted and tacitly assumed. This might be confusing to the students but it is fully understood by the instructors.

We wanted to remind our reader of this simple guiding rule for introductory physics—and, in fact, for all academic courses where we pass structured information to students. Nonetheless, we will ignore it from now on since our article is not an attack against the authors' motivation for the writing of article [1]. Instead, we will agree with the authors that, to reduce the confusion that may occur to some students who realize any of the subtleties that exist in the models we use in introductory physics, we should search to construct simple models whose solutions by elementary methods are exact. Unfortunately, it is easier to say this than to execute it. In most cases, such constructions are really hard. However, in this paper, we do two things. First, we address the conical resistor introduced by Romano and Price in article [1], and provide a derivation for its resistance which was omitted by the authors. Then, we present a modification of the truncated cone resistor that can be solved easily by elementary methods in introductory physics. Furthermore, at the same time, it can be solved easily by advanced methods in more advanced courses to demonstrate other mathematical techniques in electricity.

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notice how some texts by the same authors in our references have dropped the resistor in newer editions.

2 About the truncated cone resistor

In the chapter[2] where electric current and ohmic materials are introduced, introductory physics texts [4, 5] derive the formula for the electric resistance R encountered by a uniform current running along the axis of a uniform cylindrical wire made of a material with resistivity ρ. That formula is

$$R = \rho \frac{\ell}{A}. \tag{1}$$

In this formula, ℓ is the length of the wire, and A is its cross-sectional area. Incidentally, note that the cylinder does not have to be a circular cylinder: the cross-section may be of any shape. In deriving this formula in an introductory physics course, the assumption of uniformity is assumed to be straightforward and never discussed thoroughly. Work has been done to study the distribution of the surface charges on the wire needed to produce a uniform field inside the wire [6]. Although the study of surface charges gives an insight into the physics of what is happening inside the wire, it is beyond the scope of our interest. We are interested solely in the calculation of resistance!

Needless to say, a major part of the students' experiences in an introductory physics course is to understand how the formulæ derived under uniform conditions can be used when non-uniformities arise. To demonstrate to students how formula (1) can be applied to non-uniform resistors when the non-uniformity arises from their geometry, many texts [4] include as an end-of-chapter problem the truncated conical resistor (seen in Figure 1).

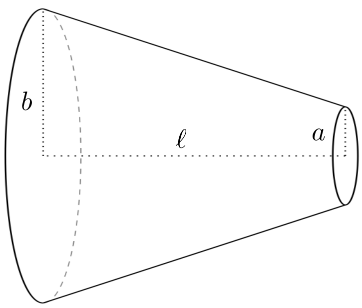

Figure 1: A resistor in the form of a truncated cone.

The expected method for the solution is slicing the resistor into coin-like infinitesimal resistors connected in series, and then calculate the total resistance by adding the infinitesimal resistance of all such infinitesimal resistors. This method of addition of electrical resistances (as well as other quantities) has been discussed extensively in the articles [2, 3]. The interested

[2] This chapter is usually entitled *electric current* or something similar.

reader should consult them for obtaining a more rounded understanding of the technique.

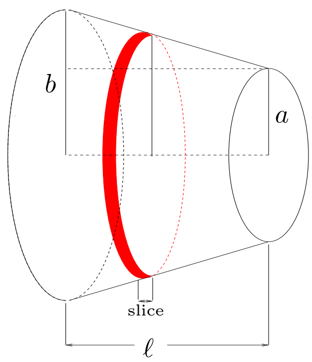

Figure 2: The slicing of the truncated cone to infinitesimal resistors.

Once the method has been applied, the resistance of the truncated cone is found to be

$$R = \rho \frac{\ell}{\pi\, a\, b}, \tag{2}$$

where ℓ is the height of the truncated cone and a, b the radii of the bases. Result (2) appears to be reasonable. It reduces to that of a uniform cylinder if $a = b$. However, as has already been mentioned, the expected method of solution is plagued by deficiencies. Although a requirement for the calculation, the planar surfaces that define the infinitesimal resistors cannot be truly equipotential surfaces. Starting from this point and discussing the implications, Romano and Price (a) attempt to correct the result by providing a numerical solution and (b) suggest a model in which the result (2) is correct. In particular, they suggest that for a resistor made of an electrical biresistive[3] material with resistivity $\rho_l = \rho$ along the axis and resistivity $\rho_t = 0$ transversely to the axis, the result (2) is indeed correct. Biresistivity is not a bad idea given the ingenuity of the materials created by material scientists. After all, we can imagine the conical resistor made of graphite which consists of a stack of sheets; current in graphite flows very easily within each sheet, but it is much harder to flow from sheet to sheet.

In this paper, we will follow a different approach. We will ask the question as follows. In particular: *Can we create a modified truncated cone for which the expected solution method of introductory physics provides the correct result?* And the result should not be correct by accident but because the method is fundamentally correct. The answer to this question is affirmative. Here is the new resistor: Starting with a solid cylinder of radius r_2, we remove a

[3]To the best of our knowledge, this is not an existing term. We made it up in analogy to optical birefringent materials.

coaxial cylinder of radius r_1, where $r_1 < r_2$. We then cut our wedge out of this cylinder by slicing along two of its radii. The resulting shape is shown in Figure 3, and we will call it the truncated wedge.

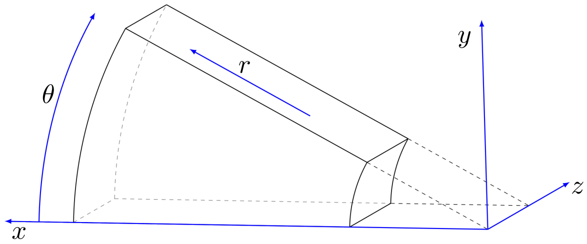

Figure 3: A resistor in the form of a truncated wedge (cut from a hollow circular cylinder). The figure also shows the various coordinates we use for its description. Besides the Cartesian coordinates x, y, z, since, mathematically, the resistor is a right cylinder, cylindrical coordinates r, θ, z are a natural choice. The bases of the cylinder are curved trapezoids at $z = 0$ and $z = a$. The parallel sides of the trapezoids are the curved ones given by $r = r_1$ and $r = r_2$ (with $0 \leq \theta \leq \theta_0$). The resistor can be seen as a stack of layers (trapezoids). When the physics is identical in each layer (i.e. there is translational symmetry along the z-directions), the problem is effectively 2-dimensional on the xy-plane. In this case, complex analysis becomes a powerful tool. We will indicate by ζ the complex coordinate of the complex xy-plane: $\zeta = x + i\,y = r(\cos\theta + i\,\sin\theta) = r\,e^{i\theta}$.

A resistor in the form of a truncated wedge can be given as an easy application of formula (1) in introductory physics. At the same time, as we will discuss, the geometry of the resistor manages to overcome all deficiencies in the original conical resistor.

3 Conical resistor truncated by spherical shells

In this section, we will address the conical resistor introduced by Romano and Price in paper [1]. Moreover, since the authors omitted the derivation of the resistance of the conical resistors we will provide the reader with a two methods of deriving its resistance. First, we use the method of adding infinitesimal resistors thought in introductory physics courses. Following that, we apply the techniques form advanced electromagnetism and solve the Laplace equation in spherical coordinates to compute the resistance of a cone using the boundary conditions. This solution can be used as an example in a Mathematical Methods course where Fourier analysis and special functions are introduced or in the advanced Electromagnetism course where boundary value problems are routinely solved.

We ask the reader to note that both methods yield the same solution, which was also provided as the result in paper [1].

3.1 Introductory physics method: adding infinitesimal resistors

To construct the same resistor as described in paper [1], we start with the cone. Then, we truncate it using two spheres of radii $\tilde{R}$ and $\tilde{R}+\tilde{L}$ as shown in Figure 4. We are left with the conical resistor portrayed in Figure 5.

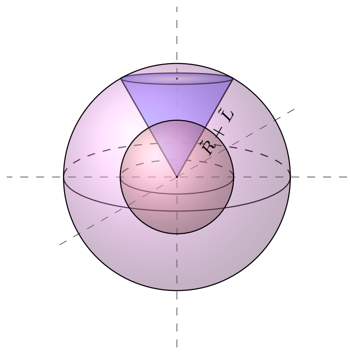

Figure 4: Truncating the cone with spherical shells of radii $\tilde{R}$ and $\tilde{R}+\tilde{L}$

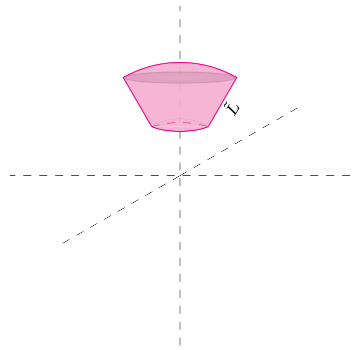

Figure 5: The leftover truncated cone

Here is the look at the cross-section of our conical resistor.

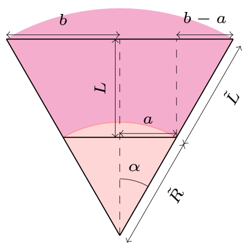

Figure 6: The cross-section of the cone

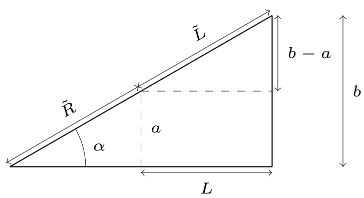

Figure 7: Equilateral triangle (half of the cross-section of the cone)

3.1.1 Useful identities

$$\tilde{L} = \sqrt{L^2 + (b-a)^2} = \frac{b-a}{sin(\alpha)} \tag{3}$$

$$\tilde{R} = \frac{a}{sin(\alpha)} \tag{4}$$

$$\tilde{R} + \tilde{L} = \frac{b}{sin(\alpha)} \tag{5}$$

$$\sin(\alpha) = \frac{b-a}{\tilde{L}} = \frac{b-a}{\sqrt{L^2+(b-a)^2}} \tag{6}$$

$$\cos(\alpha) = \frac{L}{\tilde{L}} = \frac{L}{\sqrt{L^2+(b-a)^2}} \tag{7}$$

$$\tan(\alpha) = \frac{b-a}{L} \tag{8}$$

$$1-\cos(\alpha) = 1-\frac{L}{\sqrt{L^2+(b-a)^2}} = \frac{\sqrt{L^2+(b-a)^2}-L}{\sqrt{L^2+(b-a)^2}} = \frac{\tilde{L}-L}{\tilde{L}} \tag{9}$$

3.1.2 Deriving the area of the spherical cap

We split the conical resistor into an infinitesimal resistors along the radial direction r. The corresponding cross-sectional area of a spherical cap across the angle α and at a radial distance r is calculated using the spherical coordinates in the following manner.

$$\begin{aligned} S(r) &= r^2 \int_0^{2\pi} d\phi \int_0^{\alpha} d\theta \sin(\theta) \\ &= 2\pi r^2 (1-\cos(\alpha)) \\ &= 2\pi r^2 \frac{\sqrt{L^2+(b-a)^2}-L}{\sqrt{L^2+(b-a)^2}} \\ &= 2\pi r^2 \frac{\tilde{L}-L}{\tilde{L}} \end{aligned} \tag{10}$$

3.1.3 Deriving the resistance of the truncated cone

Using the area of the spherical cap that we found in Equation (10), we can now derive the resistance of the cone truncated by spherical shells. We approach the problem by summing up the resistance of all of the infinitesimal pieces. Formula for the resistance of the infinitesimal piece is $dR = \rho \frac{dr}{S(r)}$.

$$
\begin{aligned}
R &= \int dR \\
&= \frac{\rho}{2\pi \frac{\tilde{L}-L}{\tilde{L}}} \int_{\tilde{R}}^{\tilde{R}+\tilde{L}} \frac{dr}{r^2} \\
&= \frac{\rho}{2\pi} \frac{\tilde{L}}{\tilde{L}-L} \left(\frac{1}{\tilde{R}} - \frac{1}{\tilde{R}+\tilde{L}} \right) \\
&= \frac{\rho}{2\pi} \frac{\tilde{L}}{\tilde{L}-L} \left(\frac{\tilde{L}}{\tilde{R}(\tilde{R}+\tilde{L})} \right) \\
&= \frac{\rho}{2\pi} \frac{\tilde{L}^2}{\tilde{L}-L} \left(\frac{1}{\tilde{R}(\tilde{R}+\tilde{L})} \right)
\end{aligned}
$$

Now, using identities from Equations (3), (4), and (5) we obtain the following.

$$
\begin{aligned}
R &= \frac{\rho}{2\pi} \frac{\tilde{L}^2}{\tilde{L}-L} \frac{sin(\alpha)}{a} \frac{sin(\alpha)}{b} \\
&= \frac{\rho}{2\pi ab} \frac{\left(\tilde{L}\sin(\alpha)\right)^2}{\tilde{L}-L} \\
&= \frac{\rho}{2\pi ab} \frac{(b-a)^2}{\sqrt{L^2+(b-a)^2}-L}
\end{aligned} \tag{11}
$$

3.2 Advanced electromagnetism: boundary value problem in spherical harmonics

Figure 8 below, gives us another look at the geometry of the conical resistor. However, this time we present it in spherical coordinates.

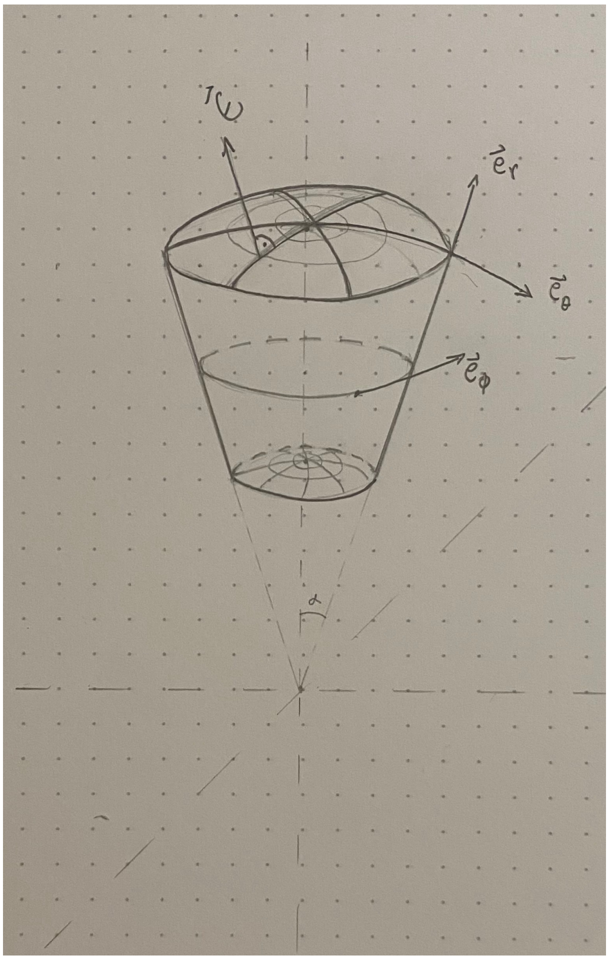

Figure 8: Conical resistor

From Figure 8, we see that:

- $\vec{E} \parallel \vec{e}_r = \frac{\vec{r}}{r}$
- $\vec{E} = \frac{\partial \Phi}{\partial r} \vec{e}_r$
- Top equipotential surface: $\Phi = 1$

- Bottom equipotential surface: $\Phi = 0$

Since we will solve this problem using spherical coordinates, we have that

$$\Phi = \Phi(r, \theta, \phi) \quad \text{and} \quad \nabla^2\Phi = 0.$$

The two curved sides of the truncated cone are equipotential surfaces so boundary conditions are as follows:

$$\Phi(\tilde{R}, \theta, \phi) = 0 \tag{BC 1}$$

$$\Phi(\tilde{R} + \tilde{L}, \theta, \phi) = 1 \tag{BC 2}$$

$$\frac{\partial \Phi}{\partial \theta} = 0 \quad \text{for all } \theta \ (0 \le \theta \le \pi) \tag{BC 3}$$

Equation (BC 3) is telling us that current does not flow parallel to $\vec{e}_{\theta}$.

Since $\nabla^2\Phi = 0$ should be satisfied for the electric potential inside the conical resistor, we will solve it in spherical coordinates.

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2(\theta)}\frac{\partial^2}{\partial \phi^2}$$

For convenience, we can write $\nabla^2\Phi$ as

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{1}{r^2}\hat{\vec{L}}^2$$

where

$$\hat{\vec{L}}^2 = -\frac{1}{\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial}{\partial \theta}\right) - \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial \phi^2}.$$

$\hat{\vec{L}}$ is the angular momentum operator from quantum mechanics and $\hat{\vec{L}}^2 = \hat{\vec{L}} \cdot \hat{\vec{L}}$.

The eigenvalue problem of $\hat{\vec{L}}^2$ is well known:

$$\hat{\vec{L}}^2 \, Y_{\ell m}(\theta, \phi) = \ell(\ell + 1) Y_{\ell m}(\theta, \phi). \tag{12}$$

Here,

$$\ell = 0, 1, 2, 3... , \quad m = 0, \pm 1, \pm 2, \pm 3... \pm \ell , \quad \text{and} \quad Y_{\ell m}(\theta, \phi)$$

are spherical harmonics, where

$$Y_{\ell m}(\theta, \phi) = C_{\ell m} e^{im\phi} \, P_{\ell}^{m} \left(\cos(\theta)\right). \tag{13}$$

$$P_\ell^m(x) = \left(1-x^2\right)^{\frac{|m|}{2}} \frac{\mathrm{d}^{|m|}}{\mathrm{d}x^{|m|}} P_\ell(x), \tag{14}$$

$$P_\ell(x) = \frac{(-1)^\ell}{2^\ell \ell!} \frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} \left(1-x^2\right)^\ell .$$

Also,

$$x = \cos(\theta)$$

and

$$C_{\ell m} = (-1)^m \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}}. \tag{15}$$

Here, $P_\ell^m(x)$ are associated Legendre polynomials and $P_\ell(x)$ are Legendre polynomials. Furthermore, ℓ-coefficients are integers because $L_2(S^2)$ is a separable space.

The eigenvalue problem from Equation (12) allows us to solve $\nabla^2\Phi(r,\theta,\phi) = 0$ with

$$\Phi(r,\theta,\phi) = R(r)Y_{\ell m}(\theta,\phi).$$

Thus, we obtain the following.

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2 \frac{\mathrm{d}}{\mathrm{d}r}\left(R(r)Y_{\ell m}(\theta,\phi)\right)\right] - \frac{1}{r^2}\hat{\vec{L}}^2 \left(R(r)Y_{\ell m}(\theta,\phi)\right) = 0,$$

and so

$$\frac{Y_{\ell m}(\theta,\phi)}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right) = \frac{R(r)}{r^2}\ \hat{\vec{L}}^2\ Y_{\ell m}(\theta,\phi)$$

since $\hat{\vec{L}}^2$ contains **only derivatives of $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$**.

Using Equation (12) we get

$$\frac{Y_{\ell m}(\theta,\phi)}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right) = \frac{R(r)}{r^2}\ell(\ell+1)\ Y_{\ell m}(\theta,\phi).$$

Furthermore, dividing both sides by RY we obtain

$$\frac{1}{R}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right) = \frac{1}{r^2}\ell(\ell+1).$$

Since we do not consider a case in which $r = 0$, we can multiply by r^2 to get

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right) = \ell(\ell+1)R. \tag{16}$$

There are **two solutions** of Equation (16):

$$R_1(r) = r^\ell \quad \text{and} \quad R_2(r) = \frac{1}{r^{\ell+1}} = r^{-(\ell+1)}$$

To see that, we compute the following.

$$\begin{aligned}\frac{\mathrm{d}R_1}{\mathrm{d}r} &= \ell r^{\ell-1} \\ r^2\frac{\mathrm{d}R_1}{\mathrm{d}r} &= \ell r^{\ell+1} \\ \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R_1}{\mathrm{d}r}\right) &= \ell(\ell+1)r^{\ell} \\ &= \ell(\ell+1)R_1(r)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}R_2}{\mathrm{d}r} &= -(\ell+1)r^{-\ell-2} \\ r^2\frac{\mathrm{d}R_2}{\mathrm{d}r} &= -(\ell+1)r^{-\ell} \\ \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R_2}{\mathrm{d}r}\right) &= \ell(\ell+1)r^{-(\ell+1)} \\ &= \ell(\ell+1)R_2(r)\end{aligned}$$

Therefore, the general solution of Equation (16) is

$$R(r) = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \equiv R_\ell(r). \tag{17}$$

So,

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} R_\ell(r)Y_{\ell m}(\theta,\phi)D_{\ell m}$$

where $D_{\ell m}$ are constants that we need to determine.

3.2.1 Applying boundary conditions

Now, we use boundary conditions. We have the following.

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} D_{\ell m}R_\ell(r)Y_{\ell m}(\theta,\phi)$$

$$\Phi(\tilde{R},\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} D_{\ell m}R_\ell(\tilde{R})Y_{\ell m}(\theta,\phi) = 0 \tag{BC 1}$$

$$\Phi(\tilde{R}+\tilde{L},\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} D_{\ell m}R_\ell(\tilde{R}+\tilde{L})Y_{\ell m}(\theta,\phi) = 1 \tag{BC 2}$$

Then, we set

$$D_{\ell m}R_\ell(\tilde{R}) \equiv D_{\ell m}^{(1)}$$

and

$$D_{\ell m}R_\ell(\tilde{R}+\tilde{L}) \equiv D_{\ell m}^{(2)}.$$

Now, we can write the following expressions.

$$0 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} D_{\ell m}^{(1)} \, Y_{\ell m}(\theta, \phi) \tag{18}$$

$$1 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} D_{\ell m}^{(2)} \, Y_{\ell m}(\theta, \phi) \tag{19}$$

For every function $f : S^2 \to \mathbb{R}$, where S^2 is the sphere, we have a Fourier expansion in spherical harmonics. So,

$$f(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_{\ell m} Y_{\ell m}(\theta, \phi),$$

where

$$C_{\ell m} = \langle Y_{\ell m}, f \rangle = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin(\theta) \; Y_{\ell m}^*(\theta, \phi) \; f(\theta, \phi).$$

Also,

$$\begin{aligned} \langle Y_{\ell m}, Y_{\ell' m'} \rangle &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin(\theta) \; Y_{\ell m}^*(\theta, \phi) \; Y_{\ell' m'}(\theta, \phi) \\ &= \delta_{\ell, \ell'} \delta_{m, m'}. \end{aligned}$$

Recalling Equations (13), (14), and (15), we deduce that

$$Y_{00}(\theta, \phi) = C_{00} = \sqrt{\frac{1}{4\pi}}. \tag{20}$$

Using the obtained Equation (20), we may rewrite Equation (19) as

$$\sqrt{4\pi} \cdot Y_{00} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} D_{\ell m}^{(2)} \, Y_{\ell m}(\theta, \phi)$$

so

$$D_{\ell m}^{(2)} = \left\langle Y_{\ell m}, \sqrt{4\pi} Y_{00} \right\rangle = \sqrt{4\pi} \cdot \delta_{\ell,0} \delta_{m,0}.$$

Therefore,

$$D_{\ell m}^{(2)} = \begin{cases} \sqrt{4\pi}, & \text{when } \ell = m = 0 \\ 0, & \text{otherwise} \end{cases}. \tag{21}$$

Since $D_{\ell m}^{(2)} = D_{\ell m} R_\ell(\tilde{R} + \tilde{L})$, we find that

$$D_{00} \cdot \left(A_0 \cdot (\tilde{R} + \tilde{L})^0 + \frac{B_0}{(\tilde{R} + \tilde{L})^{0+1}} \right) = \begin{cases} \sqrt{4\pi}, & \text{when } \ell = m = 0 \\ 0, & \text{otherwise} \end{cases}.$$

Thus,

$$D_{00} \cdot \left(A_0 + \frac{B_0}{(\tilde{R} + \tilde{L})} \right) = \sqrt{4\pi}.$$

Since $Y_{\ell m}$ are basis of $L_2(S^2)$, they are linearly independent. Therefore, Equation (18) implies that $D^{(1)}_{\ell m} = 0$ for all ℓ and m.
Hence,

$$D_{\ell m} R_\ell(\tilde{R}) = 0. \tag{22}$$

However, from Equation (21) we have that $D_{00} \neq 0$ and $D_{\ell m} = 0$ for $\ell \neq 0$ and $m \neq 0$. Thus, Equation (22) yields the following.

$$D_{00} R_0(\tilde{R}) = 0$$

$$D_{00} \left(A_0 + \frac{B_0}{\tilde{R}} \right) = 0$$

Therefore, we have

$$D_{00} \left(A_0 + \frac{B_0}{\tilde{R} + \tilde{L}} \right) = \sqrt{4\pi}$$

and

$$D_{00} \left(A_0 + \frac{B_0}{\tilde{R}} \right) = 0.$$

The solutions are

$$D_{00} = \sqrt{4\pi} \tag{23}$$

$$A_0 = \frac{\tilde{R} + \tilde{L}}{\tilde{L}} \tag{24}$$

$$B_0 = -\frac{\tilde{R}(\tilde{R} + \tilde{L})}{\tilde{L}} \tag{25}$$

so that $A_0 + \dfrac{B_0}{\tilde{R} + \tilde{L}} = 1$ and $A_0 + \dfrac{B_0}{\tilde{R}} = 0$.

Now, from Equation (21) we find that

$$D^{(2)}_{\ell m} = D_{\ell m} R_\ell(\tilde{R} + \tilde{L}) = \begin{cases} \sqrt{4\pi}, & \text{when } \ell = m = 0 \\ 0, & \text{otherwise} \end{cases},$$

or

$$D_{\ell m} = \begin{cases} \sqrt{4\pi}, & \text{when } \ell = m = 0 \\ 0, & \text{otherwise} \end{cases} \tag{26}$$

because $R_0(\tilde{R} + \tilde{L}) = A_0 + \dfrac{B_0}{\tilde{R} + \tilde{L}} = 1$.

3.2.2 Calculating resistance

Substituting expressions of constants for Φ, we have

$$\begin{aligned}\Phi(r,\theta,\phi) &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} D_{\ell m} R_\ell(r) Y_{\ell m}(\theta,\phi) \\ &= D_{00}\cdot R_0(r)\cdot Y_{00}(\theta,\phi) \\ &= \sqrt{4\pi}\cdot\left(A_0+\frac{B_0}{r}\right)\cdot\frac{1}{\sqrt{4\pi}} \\ &= A_0+\frac{B_0}{r} \\ &= \frac{\tilde{R}+\tilde{L}}{\tilde{L}}-\frac{1}{r}\cdot\frac{\tilde{R}(\tilde{R}+\tilde{L})}{\tilde{L}}. \end{aligned} \tag{27}$$

Here, expression for D_{00} was obtained from Equation (23), for $R_0(r)$ from Equation (17), and for $Y_{00}(\theta,\phi)$ from Equation (20). Furthermore, expressions for A_0 and B_0 were obtained from Equations (24) and (25) respectively.

Now, using Equation (27) we can calculate the electric field $\vec{E}$.

$$\begin{aligned}\vec{E} &= -\nabla\Phi \\ &= -\frac{\partial\Phi}{\partial r}\vec{e}_r \\ &= \frac{1}{r^2}\cdot\frac{\tilde{R}(\tilde{R}+\tilde{L})}{\tilde{L}}\,\vec{e}_r \end{aligned} \tag{28}$$

Moreover, the current density inside the resistor $\vec{j}$ is

$$\begin{aligned}\vec{j} &= \frac{1}{\rho}\vec{E} \\ &= \frac{1}{r^2}\cdot\frac{\tilde{R}(\tilde{R}+\tilde{L})}{\rho\tilde{L}}\,\vec{e}_r. \end{aligned} \tag{29}$$

The current I, at $r=\tilde{R}+\tilde{L}$ is

$$I = \int \vec{ds}\cdot\vec{j}\,\Big|_{|\vec{r}|=\tilde{R}+\tilde{L}} \quad \text{where} \quad \vec{ds} = r^2\sin(\theta)\,d\theta\,d\phi\,\vec{e}_r.$$

So,

$$\begin{aligned} I &= \int_0^{2\pi} d\phi \int_0^{\alpha} d\theta\sin(\theta)\,r^2\,\frac{1}{r^2}\cdot\frac{\tilde{R}(\tilde{R}+\tilde{L})}{\rho\tilde{L}}\,\vec{e}_r\cdot\vec{e}_r \\ &= \frac{2\pi}{\rho}\cdot\frac{\tilde{R}(\tilde{R}+\tilde{L})}{\tilde{L}}(1-\cos(\alpha)). \end{aligned} \tag{30}$$

We know that

$$V = \Phi(\tilde{R} + \tilde{L}) = 1, \tag{BC 2}$$

and so,

$$I = \frac{V}{R} = \frac{1}{R}.$$

Hence,

$$\begin{aligned} R &= \frac{1}{I} \\ &= \frac{\rho}{2\pi} \cdot \frac{\tilde{L}}{\tilde{R}} \cdot \frac{1}{\tilde{R} + \tilde{L}} \cdot \frac{1}{(1 - \cos(\alpha))}. \end{aligned} \tag{31}$$

Substituting identity from Equation (9) into Equation (31) we get

$$\begin{aligned} R &= \frac{\rho}{2\pi} \cdot \frac{\tilde{L}}{\tilde{R}} \cdot \frac{1}{\tilde{R} + \tilde{L}} \cdot \frac{\tilde{L}}{\tilde{L} - L} \\ &= \frac{\rho}{2\pi} \cdot \frac{\tilde{L}^2}{\tilde{R}} \cdot \frac{1}{\tilde{R} + \tilde{L}} \cdot \frac{1}{\tilde{L} - L}. \end{aligned} \tag{32}$$

Continuing with substitutions of identities form Equations (3), (4), and (5) into Equation (32), we obtain that

$$R = \frac{\rho}{2\pi} \cdot \frac{(b-a)^2}{\sin^2(\alpha)} \cdot \frac{\sin(\alpha)}{a} \cdot \frac{\sin(\alpha)}{b} \cdot \frac{1}{\sqrt{L^2 + (b-a)^2} - L}.$$

Therefore, we can finally obtain the expression for the resistance of the conical resistor.

$$R = \frac{\rho}{2\pi ab} \frac{(b-a)^2}{\sqrt{L^2 + (b-a)^2} - L} \tag{33}$$

The result obtained in Equation (33) is identical to the result obtained in Equation (11)!

4 The resistance of the truncated wedge

In this section, we will work out the solution of the truncated wedge resistor using three different methods. The first method is the standard method of slicing the resistor into infinitesimal resistors as expected by students in the introductory physics course.

The second method uses conformal mapping. Complex analysis is a valuable tool for theoretical physics, with conformal mapping being one of the most important techniques within it. The method of conformal mapping and how to apply it in electrostatics situations is explained in numerous texts. (For example, see [7]). This solution is ideal to be presented in a *Mathematical*

Methods course for physicists. In such a course, it is traditional to teach a fast overview of complex analysis with various applications. Often applications to electrostatics are omitted. However, this problem can rectify this omission. The required mapping is a basic one and hence there is no need for an extensive discussion of additional concepts and constructions.

The third method uses a boundary value problem to compute the form of the potential inside the resistor. Relying on partial differential equations and being a universal method for solving problems in physics, boundary value problems are taught and used almost in all advanced courses of physics. In particular, they are used in any advanced electromagnetism course for undergraduate [8] and graduate students [9]. Hence, this solution can be used as an example in a *Mathematical Methods* course where Fourier analysis and special functions are introduced or in the advanced *Electromagnetism* course where boundary value problems are routinely solved.

An important comment is that all methods give the same result. Hence, for the intuitive physicist, no method has a preferred advantage over the other. Which solution is preferred is ultimately a matter of taste or a choice about the tool and concept to be taught. However, for the strict and more formal physicist—a mathematician's sibling—the last method is more precise. The first two solutions make use of explicit assumptions for the flow of the current. In the third solution, there is no assumption. The result follows naturally through computation. Perhaps, this is another benefit of this resistor: Make students appreciate a physicist's solution that uses shortcuts based on learned physical behavior compared to a mathematician's solution which requires no shortcut.

Before we move to the solutions, we introduce our notation. Figure 3 shows the coordinates used in the solutions: the Cartesian coordinates x, y, z; the cylindrical coordinates r, θ, z; and the complex variable $\zeta = x + iy = r\,e^{i\theta}$ on the xy-plane when the latter is seen as a complex plane.

4.1 Introductory physics: adding infinitesimal resistors

The method of splitting a non-uniform object into infinitesimal slices is one of the most powerful ideas in the history of science. It is used consistently at any level of work with increasing sophistication. For this reason, it is required that any student masters the technique as early as possible. The reader is encouraged to read References [2, 3] where the technique is explained and applied to many similar situations so the reader can obtain a solid understanding.

We split the truncated wedge into an infinitesimal resistors along the radial direction r. A representative infinitesimal resistor at a radial distance r is seen in Figure 9.

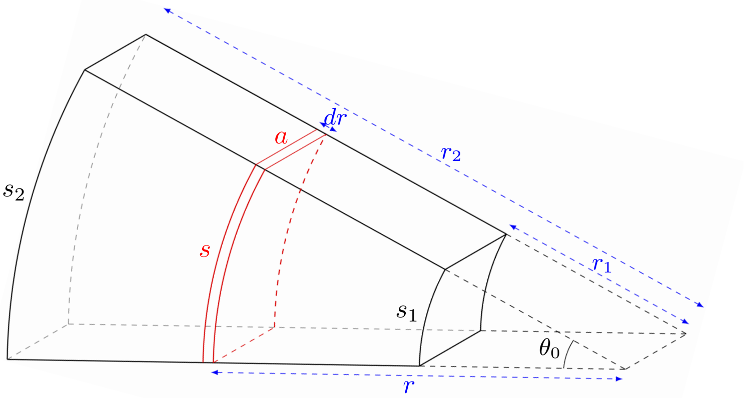

Figure 9: The truncated wedge split in infinitesimal resistors.

The length run by the current in the infinitesimal resistor is dr. The corresponding cross-sectional area is $A = sa$, with $s = r\,\theta_0$. The infinitesimal resistor is uniform and has a cylindrical shape. Hence, from equation (1), its infinitesimal resistance is

$$\mathrm{d}R = \rho\,\frac{dr}{A} = \rho\frac{dr}{a\theta_0 r}.$$

One of the subtleties of the original truncated cone is that the infinitesimal resistors are not in series and, hence, their resistances should not be added. However, in the truncated wedge case, the circular faces of the entire resistor, as well as the faces of the infinitesimal resistors are equipotential since they are perpendicular to the flow of the current. Since the infinitesimal resistors are connected in series, the resistance of the resistor is the sum of the infinitesimal resistances:

$$R = \int dR = \frac{\rho}{a\,\theta_0}\int_{r_1}^{r_2}\frac{dr}{r} = \frac{\rho}{a\,\theta_0}\ln\frac{r_2}{r_1}. \tag{34}$$

4.2 Mathematical methods: conformal mapping

When a problem has translational symmetry, it is effectively a 2-dimensional problem, and a vast array of tools can be used. In particular, complex analysis provides some of the most powerful techniques for theoretical physics. Among them, the conformal mapping of a domain to another domain can be used to transform a hard problem into a simpler one. In particular, the logarithmic map from the complex ζ-plane to the w-plane,

$$w = \ln\zeta$$

is of interest to us since it maps an annulus to a rectangle.

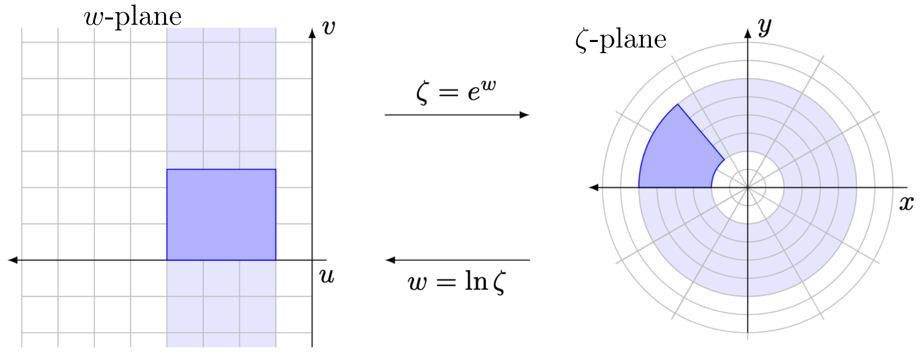

Figure 10: Conformal mapping of the planes using the exponential/logarithmic function. The points $\zeta = r\, e^{i\theta}$, $\theta \in \mathbb{R}$ of a circle of radius r on the ζ-plane map to the line $u = \ln r$ of the w-plane. In particular, if θ takes values in the interval $[0, 2\pi)$, then the circle maps to the segment with $v \in [0, 2\pi)$. Therefore, a truncated angular sector of angle θ_0 and radii r_1 and r_2 maps to the rectangle defined by the lines $u_1 = \ln r_1$, $u_2 = \ln r_2$, $v_1 = 0$, $v_2 = \theta_0$.

If $\zeta = re^{i\theta}$ and $w = u + iv$, then

$$u = \ln r, \quad v = \theta.$$

Notice that an annulus with radii r_1 and r_2, $0 \leq \theta < 2\pi$, maps to the strip on the w-plane confined by $u_1 = \ln r_1$ and $u_2 = \ln r_2$. Besides, if we constrain θ to the values in the interval $[0, \theta_0]$, that is, if we confine ourselves to a truncated circular sector on the ζ-plane, then the image is a rectangle from $v = 0$ to $v = \theta_0$. Here, the equipotential circular curves in the ζ-plane are mapped to lines parallel to the v axis in the w-plane [**?**].

We now view the truncated resistor as a series of truncated circular sectors stack on top of each other. We will assume that the current does not flow from one sector to another. This assumption makes the problem 2-dimensional. A priori it does not have to be right, but as the derivation in the next subsection proves, it is actually exact. Incidentally, note that we did assume this in the first solution two by considering radial flow (on a plane) in the first solution.

Using conformal mapping, the truncated wedge transforms into a rectangular prism.

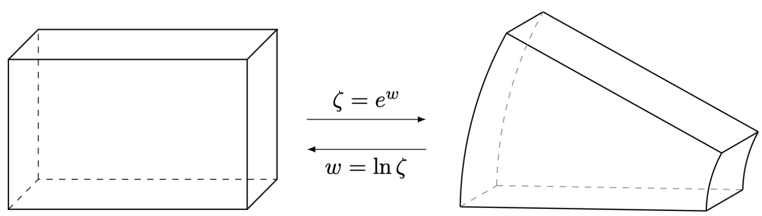

Figure 11: The logarithmic mapping maps the truncated wedge to a uniform rectangular cylinder.

As such, its resistance can be computed straightforwardly from formula (1), with length $\ell = u_2 - u_2$ and cross sectional area $A = (v_2 - v_1)a$:

$$R = \rho \, \frac{u_2 - u_1}{a(v_2 - v_1)}.$$

Returning to the original variables,

$$R = \frac{\rho}{a\,\theta_0} \, \ln \frac{r_2}{r_1},$$

which is exactly equation (34).

4.3 Advanced electromagnetism: boundary value problem

A current I in a wire flows due to an electric field $\vec{E} = -\nabla\Phi$. The relation between the two quantities is given by what is usually (incorrectly[4]) called the microscopic Ohm's law:

$$\vec{\jmath} = \frac{1}{\rho}\,\vec{E}. \tag{35}$$

The wire is overall neutral; hence, $\nabla \cdot \vec{E} = 0$ or $\nabla^2\Phi = 0$. Therefore, this equation should be satisfied for the electric potential inside the truncated wedge. Due to the 'cylindrical symmetry', we will solve it in cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = 0. \tag{36}$$

By allowing all three coordinates r, ϕ, z to appear, we make no assumption for the flow of the current inside the resistor. That is, a priori we allow the current to flow in any pattern with respect to θ and z. As we will see though, the boundary conditions will enforce the solution to be z-independent and θ-independent naturally.

The boundary conditions that we must impose are as follows:

- The two curved sides of the truncated cone are equipotential surfaces:

  $$\Phi(r_1, \theta, z) = 0, \quad \Phi(r_2, \theta, z) = 1.$$

  These are Dirichlet boundary conditions.

- The current cannot flow out of the wire on any of the faces of the lateral surface. This requires that the current flows parallel to these faces—it has no component perpendicular to them—when computed on the lateral surface of the truncated cone. Stated mathematically:

  $$\frac{\partial\Phi(r,\theta,0)}{\partial z} = \frac{\partial\Phi(r,\theta,a)}{\partial z} = \frac{\partial\Phi(r,0,z)}{\partial\theta} = \frac{\partial\Phi(r,\theta_0,z)}{\partial\theta} = 0.$$

  These are Neumann boundary conditions.

[4]Strictly speaking, Ohm's law is the statement that *in most materials, ρ is constant.* However, (35) and $V = IR$ are known as Ohm's laws since ρ appears in them, (in the first equation explicitly and in the second through equation (1)).

The boundary problem we must solve has mixed (Dirichlet and Neumann) boundary conditions; such a problem is known as a Robin problem.
The procedure to solve the problem is standard: using the separation of variables. That is, we search for solutions

$$\Phi(r, \theta, z) = R(r)\Theta(\theta)Z(z), \tag{37}$$

which satisfy the homogeneous boundary conditions.
Inserting (37) in (36):

$$\frac{1}{rR}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{1}{r^2\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} + \frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = 0$$

or

$$\frac{1}{rR}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{1}{r^2\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} = -\frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2}.$$

Since the two sides depend on different variables, each must be a constant; let's indicate it by λ^2. Hence,

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} + \lambda^2 Z = 0, \tag{38a}$$

$$\frac{r}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \lambda^2 r^2 = -\frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2}. \tag{38b}$$

The solution for (38a) is

$$Z(z) = A\cos(\lambda z) + B\sin(\lambda z) \quad \text{or} \quad Z(z) = A + B\,z.$$

Applying the boundary conditions at $z = 0$ and $z = a$, we find that

$$B = 0, \quad \lambda = \frac{n\pi}{a}, \quad n = 1, 2, \ldots$$

Notice that the negative values of the integers do not give additional solutions. Hence,

$$Z_n(z) = A\cos\left(\frac{n\pi z}{a}\right), \quad n = 0, 1, 2, \ldots$$

Now, we look at the equation (38b),

$$\frac{r}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \frac{n^2\pi^2}{a^2}r^2 = -\frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2},$$

where we have substituted λ by the values we found above. Since the two sides depend on different variables, each must be a constant; let's indicate it by μ^2. Hence

$$\frac{r}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \frac{n^2\pi^2}{a^2}r^2 = \mu^2 \tag{39a}$$

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} + \mu^2\,\Theta = 0. \tag{39b}$$

Equation (39b) can be solved quickly:

$$\Theta(\theta) = C\cos(\mu\theta) + D\sin(\mu\theta) \quad \text{or} \quad \Theta(\theta) = C + D\,\theta.$$

We can also apply the boundary conditions at $\theta = 0$ and $\theta = \theta_0$ to find

$$D = 0, \quad \mu = \frac{m\pi}{\theta_0}, \quad m = 1, 2, \ldots$$

Again, the negative values of the integers do not give additional solutions. Hence,

$$\Theta(z) = C\cos\left(\frac{m\pi\theta}{\theta_0}\right), \quad m = 0, 1, 2, \ldots$$

Finally, equation (39a) is

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) R = 0, \quad \text{if } m = n = 0,$$

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \left(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{\theta_0^2 r^2}\right) R = 0, \quad \text{if } |n| + |m| \neq 0.$$

The solution to the former equation is

$$R(r) = G\,\ln r + F,$$

while the solution to the latter equation can be written in terms of the modified Bessel functions[5] as follows:

$$R(r) = G\,I_{\frac{m\pi}{\theta_0}}\left(\frac{n\pi r}{a}\right) + F\,K_{\frac{m\pi}{\theta_0}}\left(\frac{n\pi r}{a}\right).$$

Applying the boundary condition at $r = r_1$, we have

$$G = -F\,\frac{1}{\ln r_1}, \quad G = -F\,\frac{K_{\frac{m\pi}{\theta_0}}\left(\frac{n\pi r_1}{a}\right)}{I_{\frac{m\pi}{\theta_0}}\left(\frac{n\pi r_1}{a}\right)},$$

respectively. Therefore, the most general solution before implementing the last boundary condition is

$$\Phi(r, \theta, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E_{nm}\cos\left(\frac{n\pi z}{a}\right)\cos\left(\frac{m\pi\theta}{\theta_0}\right) R_{nm}(r),$$

where

$$R_{00}(r) = 1 - \frac{\ln r}{\ln r_1},$$

$$R_{nm}(r) = \left[K_{\frac{m\pi}{\theta_0}}\left(\frac{n\pi r}{a}\right) - \frac{K_{\frac{m\pi}{\theta_0}}\left(\frac{n\pi r_1}{a}\right)}{I_{\frac{m\pi}{\theta_0}}\left(\frac{n\pi r_1}{a}\right)} I_{\frac{m\pi}{\theta_0}}\left(\frac{n\pi r}{a}\right)\right], \quad |n| + |m| \neq 0,$$

[5]When the index of the Bessel functions is not an integer, it is more precise to call them cylinder functions.

and E_{nm} are some constants. We can now impose the second Dirichlet boundary condition at $r = r_2$:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \tilde{E}_{nm} \cos\left(\frac{n\pi z}{a}\right) \cos\left(\frac{m\pi\theta}{\theta_0}\right) = 1, \tag{40}$$

where we have set

$$\tilde{E}_{00} = E_{00}\left(1 - \frac{\ln r_2}{\ln r_1}\right),$$

$$\tilde{E}_{nm} = E_{nm}\left[K_{\frac{m\pi}{\theta_0}}\left(\frac{n\pi r_2}{a}\right) - \frac{K_{\frac{m\pi}{\theta_0}}\left(\frac{n\pi r_1}{a}\right)}{I_{\frac{m\pi}{\theta_0}}\left(\frac{n\pi r_1}{a}\right)} I_{\frac{m\pi}{\theta_0}}\left(\frac{n\pi r_2}{a}\right)\right], \quad |n| + |m| \neq 0.$$

Equation (40) is a double Fourier series with Fourier coefficients $\tilde{E}_{nm}$. The latter can be computed by

$$\tilde{E}_{nm} = \left(\frac{2}{a}\int_0^a \cos\left(\frac{n\pi z}{a}\right) dz\right)\left(\frac{2}{\theta_0}\int_0^{\theta_0} \cos\left(\frac{m\pi\theta}{\theta_0}\right) d\theta\right) = 0, \quad if\ |n| + |m| \neq 0.$$

This implies that $E_{nm} = 0$ unless $n = m = 0$. Then $\tilde{E}_{00} = 1$ which gives

$$E_{00} = \frac{\ln r_1}{\ln \frac{r_1}{r_2}}.$$

The solution for the potential is thus

$$\Phi(r, \phi, z) = \frac{\ln \frac{r}{r_1}}{\ln \frac{r_2}{r_1}}.$$

From this, the magnitude of the current density is found to be

$$j(r) = \frac{1}{\rho\, r\, \ln \frac{r_2}{r_1}}.$$

Notice that, for any fixed r, this density is uniform—it has the same value over the cross-sectional area of the resistor. Of course, this statement is nothing new given all the comments which have been made up to now. At $r = r_1$, the current density is $j = \frac{1}{\rho\, r_1\, \ln \frac{r_2}{r_1}}$. Hence,

$$I = j\, s_1 a = \frac{a\theta_0}{\rho\, \ln \frac{r_2}{r_1}},$$

and, from the macroscopic Ohm's law, $R = V/I$,

$$R = \frac{\rho}{a\,\theta_0}\, \ln \frac{r_2}{r_1}.$$

Once more, the result we found previously.

5 Discussion and Conclusion

Using three different methods from introductory physics, complex analysis and the boundary value problem, we calculated the resistance of a modified conical wedge Figure 3. All three methods used gave the same form of total resistance. We believe the final solution we get is exact and in the form:

$$R = \frac{\rho}{a\,\theta_0} \ln \frac{r_2}{r_1}.$$

Notice here that if $\theta_0 = 2\pi$, then the resistance will be $R = \frac{\rho}{2\pi a} \ln \frac{r_2}{r_1}$, which is a well-known result of the resistance of a hollow cylinder. This shows that the shape we are introducing in this paper is actually a cut from a cylinder with some angle θ_0, inner and outer radii r_1, r_2.

We introduced this shape to overcome inaccuracies faced in the calculation of the conical resistance using the Riemann sum method. In principle, the result obtained using the introductory physics method does not have to be exact since it assumes that the infinitesimal slabs connected in series are equipotential surfaces. This does not have to be the case! However, in this modified problem, the plausible choice of the equipotential surfaces allows for the use of the Riemann sum of slabs causing no trouble with the physics behind. This argument suggests that the modified conical problem we are introducing is better to be used in introductory physics books instead of the regular cone problem. Moreover, further study should be done to find the exact resistance of the regular conical shape.

Hence, introductory physics texts should use the truncated wedge as a training problem for students. Not only it can be solved easily by the elementary method of the Riemann integral, but the result is exact. It can be used to introduce students to symmetries, to show them the relation between the flow of the current and equipotential lines, and to demonstrate other ideas which they learn in the introductory course. Furthermore, they can revisit the problem in more advanced classes and use it again as a training tool for conformal mapping and boundary value problems. In this way, they have a standard to compare and evaluate techniques.

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